1 Maths

1.1 Tensors

- Roughly speaking, tensors are quantities with indices that transform in the appropriate way under a change of coordinates.

- A vector **v** is an example of a (1,0) tensor. If it has components v^α with respect to coordinates x^α and components v'^α with respect to coordinates x'^α, then

$$v'^\alpha = v^\beta \frac{\partial x'^\alpha}{\partial x^\beta}\,, \quad v^\alpha = v'^\beta \frac{\partial x^\alpha}{\partial x'^\beta} \tag{1}$$

- A more geometrical definition of a vector is as a tangent to a curve passing through a point, or as a directional derivative operator that acts on functions. With this latter interpretation the coordinate basis vectors $\mathbf{e}_\alpha$ are simply directional derivatives along the coordinate lines x^α holding all other coordinates fixed: $\mathbf{e}_\alpha = \partial/\partial x^\alpha$.

- A dual vector ω is an example of a (0,1) tensor. If it has components ω_α with respect to coordinates x^α and components ω'_α with respect to coordinates x'^α, then

$$\omega'_\alpha = \omega_\beta \frac{\partial x^\beta}{\partial x'^\alpha}\,, \quad \omega_\alpha = \omega'_\beta \frac{\partial x'^\beta}{\partial x^\alpha} \tag{2}$$

- The canonical example of a dual vector ω is the gradient of a function: $\omega_\alpha = \partial f/\partial x^\alpha$.

- The dual basis vectors $\mathbf{e}^\alpha$ are simply coordinate differentials: $\mathbf{e}^\alpha = \mathbf{d}x^\alpha$.

- The metric tensor **g** that appears in a line element is an example of a (0,2) tensor. If $g_{\alpha\beta}$ are the components of **g** with respect to coordinates x^α and $g'_{\alpha\beta}$ are the components of **g** with respect to coordinates x'^α, then

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}\,, \quad g_{\alpha\beta} = g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \tag{3}$$

- The metric tensor is symmetric under interchange of its indices: $g_{\alpha\beta} = g_{\beta\alpha}$. It is also invertible in the sense that the matrix inverse of $g_{\alpha\beta}$ (denoted $g^{\alpha\beta}$) exists. It satisfies

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma \tag{4}$$

- The metric tensor (and its inverse) can be used to identify vectors and dual vectors, and to raise and lower indices on arbitrary tensors. The correspondence between a vector and dual vector defined via

$$v_\alpha := g_{\alpha\beta} v^\beta \tag{5}$$

  is called *lowering of an index with the metric.* Similarly, given the components ω_β of a dual vector, we can define a vector via

$$\omega^\alpha := g^{\alpha\beta} \omega_\beta \tag{6}$$

  This operation is called *raising of an index with the metric.*

- If the components of a tensor vanish with respect to one basis, they vanish with respect to *any other basis*. Hence if one can show the equality of the components of two tensors in one basis, then the two tensors are equal.

- Not all quantities with indices are tensors. The coordinates x^α are not the components of a vector, nor are the Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$ the components of a (1,2) tensor. (More on this later.)

1.2 Geodesic equation

- Geodesics are the straightest possible lines in a curved space. They minimize distance in flat 2-d space and maximize proper time in 4-d Minkowski spacetime.

- Since a free particle moves on a timelike geodesic, its equation of motion can be obtained by extremising the proper time integral between two timelike-separated events. Thus, the geodesic equation is simply the E-L equations for the Lagrangian

  $$L\left(x^\alpha(\sigma), \frac{dx^\alpha(\sigma)}{d\sigma}, \sigma\right) := \frac{d\tau}{d\sigma} = \sqrt{-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} \tag{7}$$

  The result being

  $$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{8}$$

  where

  $$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}\left(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\beta\gamma}\right) \tag{9}$$

  are the *Christoffel symbols*, and $g^{\alpha\beta}$ are the components of the inverse matrix to $g_{\alpha\beta}$.

- The Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$ are symmetric with respect to interchange of β and γ. In 4-d, there are 40 independent Christoffel symbols; in 3-d, 18; in 2-d, 6.

- Probably the simplest (and fastest) way to calculate the Christoffel symbols given a metric is to explicitly write down the E-L equations for the Lagrangian (7) and then identify the Christoffel symbols by comparing the equations for each coordinate x^α with the geodesic equation (8).

1.3 Covariant differentiation

- Covariant differentiation is a way of differentiating a tensor field that results in a quantity which transforms properly under coordinate transformations.

- Since a general curved spacetime is *locally* flat, one can define covariant derivative as in flat spacetime, provided the definition is valid in *arbitrary curvilinear coordinates.*

- The key difference between curvilinear and rectilinear (or Cartesian) coordinates is that the curvilinear coordinate basis vectors change, in general, from point to point—e.g., the coordinate basis vectors in flat 2-space in plane polar coordinates (r, ϕ).

- *Thus, when differentiating vector fields in arbitrary curvilinear coordinates, it is necessary to differentiate the coordinate basis vectors in addition to differentiating the vector components.*

- The derivatives of the coordinate basis vectors define the Christoffel symbols ${\Gamma^\mu}_{\alpha\beta}$

  $$\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha =: {\Gamma^\mu}_{\alpha\beta}\,\mathbf{e}_\mu \tag{10}$$

  which we've seen earlier in the context of the geodesic equation.

- Note that the Christoffel symbols are equal to zero for Cartesian coordinates since the coordinate basis vectors are constant. (Hence the Christoffel symbols are *not* the components of a $(1,2)$ tensor because they equal zero with respect to some set of coordinates and non-zero with respect to others.)

- In terms of Christoffel symbols, the derivative of a vector field can be written in arbitrary coordinates x^α as:

  $$\nabla_{\mathbf{e}_\alpha}\mathbf{v} = \nabla_{\mathbf{e}_\alpha}(v^\beta\mathbf{e}_\beta) = \left(\partial_\alpha v^\beta\right)\mathbf{e}_\beta + v^\beta\,\nabla_{\mathbf{e}_\alpha}\mathbf{e}_\beta = \left(\partial_\alpha v^\mu + {\Gamma^\mu}_{\beta\alpha}v^\beta\right)\mathbf{e}_\mu \tag{11}$$

- The above quantity in parentheses are the components of the covariant derivative $\nabla\mathbf{v}$:

  $$\nabla_\alpha v^\mu := \partial_\alpha v^\mu + {\Gamma^\mu}_{\beta\alpha}v^\beta \tag{12}$$

- One can show that the geodesic equation can be written as

  $$u^\alpha\nabla_\alpha u^\beta = 0 \tag{13}$$

  or, equivalently, $\nabla_{\mathbf{u}}\mathbf{u} = 0$. Since the acceleration vector $\mathbf{a}$ is given by $\mathbf{a} = \nabla_{\mathbf{u}}\mathbf{u}$, geodesic motion corresponds to zero acceleration.

- The covariant derivative of a scalar field ϕ is just the gradient:

  $$\nabla_\alpha f = \frac{\partial f}{\partial x^\alpha} \equiv \partial_\alpha f \tag{14}$$

- Covariant derivative obeys the product rule—e.g.,

  $$\nabla_\alpha(\omega_\beta v^\beta) = (\nabla_\alpha\omega_\beta)v^\beta + \omega_\beta(\nabla_\alpha v^\beta) \tag{15}$$

  since ordinary partial derivative obeys the product rule and covariant derivative is just partial derivative in Cartesian coordinates.

- The covariant derivative of a dual vector field ω can be found by setting $f = \omega_\alpha v^\alpha$ and using the previous results for the covariant derivative of a vector and scalar field. The result is:

  $$\nabla_\alpha\omega_\beta = \partial_\alpha\omega_\beta - {\Gamma^\mu}_{\beta\alpha}\omega_\mu \tag{16}$$

  Note the minus sign in front of the Christoffel symbol in the above equation.

- More generally,

  $$\nabla_\alpha {T^\beta}_{\mu\nu} = \partial_\alpha {T^\beta}_{\mu\nu} + {\Gamma^\beta}_{\sigma\alpha}{T^\sigma}_{\mu\nu} - {\Gamma^\sigma}_{\mu\alpha}{T^\beta}_{\sigma\nu} - {\Gamma^\sigma}_{\nu\alpha}{T^\beta}_{\mu\sigma} \tag{17}$$

- The covariant derivative of the metric is:

$$\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} - \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} \tag{18}$$

  In LICs $\partial_\mu g_{\alpha\beta} = 0$ and $\Gamma^\mu{}_{\alpha\beta} = 0$, so

$$\nabla_\mu g_{\alpha\beta} = 0 \tag{19}$$

  in these coordinates. But since $\nabla_\mu g_{\alpha\beta} = 0$ is a valid tensor equation, it is true in *all* coordinate systems.

- Given $\nabla_\mu g_{\alpha\beta} = 0$, one can show

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) \tag{20}$$

  which is the usual expression for the Christoffel symbols in terms of first partial derivatives of the metric components.

1.4 Riemann curvature from geodesic deviation

- A gravitational field can be detected by monitoring the separation of two freely-falling particles.

- In Newtonian gravity, the equation of motion for a single freely-falling particle moving in a gravitational potential $\Phi = \Phi(x^k)$ is

$$\frac{d^2 x^i}{dt^2} = -\delta^{ij} \partial_j \Phi \tag{21}$$

- If a second freely-falling particle is displaced from the first by χ^i, where $|\chi^i| \ll 1$, then

$$\frac{d^2 \chi^i}{dt^2} = -\delta^{ij} \partial_j \partial_k \Phi \bigg|_0 \chi^k \tag{22}$$

  where the second partial derivative of Φ is evaluated at the location of the first particle.

- The quantity

$$\partial_j \partial_k \Phi \tag{23}$$

  is called the *Newtonian tidal acceleration tensor.*

- In GR, the *Riemann curvature tensor* plays the role of the Newtonian tidal acceleration tensor.

- To derive the form of the Riemann curvature tensor, consider two nearby geodesics parametrised by proper time τ. Let $\mathbf{u}$ denote the tangent vector to the first geodesic, and let χ denote the separation vector connecting points along the geodesics with equal values of τ. Then one can show that the acceleration of the separation of the two nearby geodesics is given by

$$a^\mu := (\nabla_u \nabla_u \chi)^\mu = R^\mu{}_{\nu\alpha\beta} u^\nu u^\alpha \chi^\beta \tag{24}$$

  where

$$R^\mu{}_{\nu\alpha\beta} := \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\lambda\alpha} \Gamma^\lambda{}_{\nu\beta} - \Gamma^\mu{}_{\lambda\beta} \Gamma^\lambda{}_{\nu\alpha} \tag{25}$$

  The equation for a^μ is called the *geodesic deviation equation.*

- The Riemann tensor thus encodes the failure of initially parallel geodesics to remain parallel. In other words, the tidal effects of gravitation can be attributed to the local curvature of spacetime itself, and not to some "mysterious" force called gravity.

1.5 Einstein tensor

- The *Einstein tensor* $G_{\alpha\beta}$ that appears on the LHS of the Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ is constructed from the Riemann curvature tensor $R^{\alpha}{}_{\beta\mu\nu}$ and the metric tensor $g_{\alpha\beta}$ as follows:

  (i) First define the *Ricci tensor*:

  $$R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta} \tag{26}$$

  This tensor is symmetric under interchange of α and β.

  (ii) Then define the *Ricci scalar* (or scalar curvature):

  $$R := g^{\alpha\beta}\, R_{\alpha\beta} \tag{27}$$

  (iii) The Einstein tensor is then

  $$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} \tag{28}$$

1.6 Example: Newtonian gravity as a geometric theory

- A solution to the Einstein equation in the weak field, slow motion limit is the *static, weak field line element*:

  $$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[dx^2 + dy^2 + dz^2] \tag{29}$$

  Here $\Phi = \Phi(\vec{x})$ is the gravitational potential, which is assumed to be small—i.e., $|\Phi| \ll 1$. Φ satisfies the standard Poisson equation $\nabla^2\Phi = 4\pi G\mu$, where μ is the (dominant) T_{tt} component of the stress-energy tensor.

- The non-relativistic equations of motion for a particle moving in response to a gravitational potential Φ can be obtained (to first-order in Φ and velocities) by extremising the proper time integral

  $$\tau_{PQ}[\vec{x}(t)] := \int_P^Q d\tau = \int_{t_P}^{t_Q} dt \sqrt{(1+2\Phi) - (1-2\Phi)|\vec{v}|^2} \approx \int_{t_P}^{t_Q} dt \left(1 - \frac{1}{2}|\vec{v}|^2 + \Phi\right) \tag{30}$$

- Note that up to the additive constant and a multiplicative factor of the particle's mass m, the action is just the integral of the standard Lagrangian for Newtonian mechanics $L = T - V = m|\vec{v}|^2/2 - m\Phi$, so the equation of motion for the particle is the usual one: $\vec{a} = -\vec{\nabla}\Phi$.

- Note that in this weak field, slow motion approximation, the Φ contribution to the spatial part of the line element can be ignored in the Lagrangian since it is multiplied by a factor of $|\vec{v}|^2$; only the Φ in the time part of the line element contributes. Hence, *non-relativistic particle motion in Newtonian gravity is simply due to the curvature of time, not space!* [N.B.: The motion of a relativistic particle (like a photon) in this framework does include the Φ contribution to the spatial part of the line element since $|\vec{v}|^2 \sim 1$.]